Damped motion of a vertical spring.

Modeling a vertical spring and its position over time is modeling a damped simple harmonic motion, where “damped” means a gradual decrease over time of the amplitude of a periodic motion due to an external force.

A simplified equation describing the stretch of the spring vs. time can be found by combining and simplifying the equations for the amplitude for damped simple harmonic motion and the simple harmonic motion position v. time equation, as described in an article by Samhitha Bodangi.

Because an external force is acting on the vertical spring, the amplitude of each oscillation will be slightly less than the amplitude of the oscillation before. The amplitude for the displacement of damped oscillations can be written as

$$x_{max} = x_{0\,max} \cdot (e^{(\frac{-t}{\mathcal{T}})})$$

This equation represents the amplitude of each oscillation, which decreases over time, with $\mathcal{T}$ being the time constant, which represents the time to lose 63% of the original displacement.

This equation can be substituted in place of the maximum amplitude in the simple harmonic motion position vs. time equation.

$$x = x_{0\,max} \cdot (e^{(\frac{-t}{\mathcal{T}})}) \cdot sin(\sqrt{\frac{k}{m}} \cdot t)$$

This equation represents the damped motion of the vertical spring, with x being the stretch of the spring, k being the spring constant, and m being the mass of the block attached to the spring (the spring itself being supposed to have a negligible mass).

If we want the user input to be m, k, and x_0, we have to rewrite the time constant in terms of these variables. The time constant $\mathcal{T}$ representing the time to lose 63% of the original stretch of the spring, we can make the following calculation:

$$e^{\frac{-t}{\mathcal{T}}} = 0.37 \rightarrow \frac{-t}{\mathcal{T}} = ln(0.37) \rightarrow \mathcal{T} = \frac{-t}{ln(0.37)}$$

We know that the period (T) of a spring-mass system is given by the equation

$$T = 2\pi\sqrt{\frac{m}{k}}$$

If we want 63% of the initial stretch distance to be maintained for at least 10 cycles, then 63% of the initial amplitude must be maintained for 10 periods. Substituting t by the formula for 10T in the time constant equation, we get:

$$\mathcal{T} = \frac{-20\pi\sqrt{\frac{m}{k}}}{ln(0.37)}$$

And finally, substituting $\mathcal{T}$ by this formula in the simple harmonic motion position vs. time equation as described at the beginning of this document, we get the simplified formula that may be used to model the damped simple harmonic motion of a vertical spring with a block of given mass m attached, with a given spring constant k, and a given initial stretch of the spring x_0:

$$x = x_0 \cdot (e^{\left(\frac{-t}{\left(\frac{-20\pi\sqrt{\frac{m}{k}}}{ln(0.37)}\right)}\right)}) \cdot sin(\sqrt{\frac{k}{m}} \cdot t)$$

<u>Note:</u>

The curve obtained by using this equation actually is for an x-axis where, when t = 0, the stretch of the spring is 0.

The application “SimpleSpring” asks the user for the mass of the block attached to the spring, the spring constant, and the initial stretch of the spring. When the button “Calculate” is pushed, the program calculates the stretch of the spring for a time entered by the user. When the button “Draw” is pushed, the program draws the curve of the stretch of the spring vs. time; the user may choose the starting and ending time values on the graph. Unless the button “Clear” is pushed, further curves are drawn on the same graph, what allows to compare the damped motion for different values of the input variables.